Wedge and Cone Flows of Viscoelastic Liquids

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Boundary-layer flows of viscoelastic liquids occur in industrial processes and in drag reduction systems for underwater and surface vehicles. The analysis of such flows is also potentially useful in connection with the development of improved phenomenological modeling of turbulent shear flows. Several investigators, including Rivlin (1957), have suggested that the Reynolds stresses should be assumed to exhibit viscoelastic response rather than the purely viscous response inherent in the classical treatments such as Prandtl's mixing-length theory. The present note uses the second-order constitutive equation of Coleman and Noll (1960) to analyze the properties of viscoelastic boundary layers on wedges and cones. Numerical results are obtained by the method of local similarity which illustrate the effect of elasticity for various flow situations.

GOVERNING EQUATIONS

The appropriate boundary-layer equations result from specializing the equations given by Davis (1967) to the case of wedge and cone flows. In terms of the dimensionless variables defined at the end of this note one obtains

$$(\partial u/\partial s) + (j/s)u + (\partial v/\partial n) = 0$$

$$\begin{aligned} u(\partial u/\partial s) + v(\partial u/\partial n) = U_eU_e' + (\partial^2 u/\partial n^2) \\ - \alpha_1(u(\partial^3 u/\partial s\partial n^2) + (\partial u/\partial s)(\partial^2 u/\partial n^2) \\ + v(\partial^3 u/\partial n^3) - (\partial u/\partial n)(\partial^2 u/\partial s\partial n) \\ - (j/s)(2u(\partial^2 u/\partial n^2) + (\partial u/\partial n)^2)) \\ - \alpha_2(j/s)(2u(\partial^2 u/\partial n^2) + (\partial u/\partial n)^2) \end{aligned} \quad (1)$$

where (1a) results from balance of mass, (1b) results from balance of momentum in the s direction, a prime denotes differentiation with respect to s, $j = 0$ for plane flow, and $j = 1$ for axisymmetric flow. In (1) the inviscid surface speed is given by

$$U_e = s^p \quad (2)$$

[see Rosenhead (1963)]. Also

$$\alpha_1 = -\mu_1 U/\eta_0 L, \qquad \alpha_2 = \mu_2 U/\eta_0 L. \quad (3)$$

These parameters vanish for a Newtonian liquid. Thermodynamic considerations require that $\mu_1 < 0$ but place no restriction on the sign of μ_2 as discussed by Markovitz and Coleman (1964).

SOLUTION PROCEDURE

The solution is facilitated by defining new variables by the following relationships:

$$\begin{aligned} s &= \xi \\ n &= (2/(p+1))^{1/2}\xi^{(1-p)/2}\eta \\ u &= \xi^p F(\xi,\eta) \\ v &= (2/(p+1))^{1/2}\xi^{-(1-p)/2}((((p+1)/2)+j) \\ &\quad V(\xi,\eta) + ((1-p)/2)\eta F(\xi,\eta)) \end{aligned} \quad (4)$$

The equations obtained by substituting (4) and (2) into (1) can be solved by a perturbation procedure for small elastic effects. This was done by Denn (1967) and Shrestha (1970). The solutions have the form of an inverse coordinate expansion. Experience with such expansions [see Davis (1967) for the flow of a second-order liquid past a flat plate and Van Dyke (1964a, b) for some Newtonian flows] indicates that unexpected logarithmic and noninteger-power terms may be required in such solutions to obtain the correct behavior of the flow variables. The form of such terms is difficult to predict in advance. To avoid this difficulty Peddieson (1971) suggested the method of local similarity. In this procedure the dependence of the dependent variables on ξ is suppressed. Applying the method to the present problem yields

$$\begin{aligned} F(\xi,\eta) &= F(\eta) \\ V(\xi,\eta) &= V(\eta) \end{aligned} \quad (5)$$

Substituting (2), (4), and (5) into (1) produces (with a prime now denoting differentiation with respect to η)

$$\begin{aligned} V' + F = 0 \\ F'' - (1+j\beta_1)VF' + \beta_2(1-F^2) \\ + \alpha_{1\xi}((\beta_3+j)F'^2 - (\beta_4+j)VF''' \\ - 2(\beta_3-j)FF'') - j\alpha_{2\xi}(2FF''+F'^2) = 0 \end{aligned} \quad (6)$$

where

$$\begin{aligned} \alpha_{1\xi} = \alpha_1/\xi^{(1-p)}, \quad \alpha_{2\xi} = \alpha_2/\xi^{(1-p)} \quad \text{and} \\ \beta_1 = 2/(p+1), \quad \beta_2 = 2p(p+1), \\ \beta_3 = (3p-1)/2, \quad \beta_4 = (p+1)/2. \end{aligned} \quad (7)$$

For stagnation-point flow ($p = 1$) $\alpha_{1\xi}$ and $\alpha_{2\xi}$ are independent of ξ and the flow is completely self-similar. The appropriate boundary conditions are

$$F(0) = 0, \quad V(0) = V_w, \quad \lim_{\eta\to\infty} F(\eta) = 1. \quad (8)$$

The coefficient of skin friction c and the displacement thickness integral δ are defined in the usual way and found to be

$$\begin{aligned} c &= F'(0) - \alpha_{1\xi}(\beta_4+j)V(0)F''(0) \\ \delta &= \int_0^\infty (1-F)\,d\eta. \end{aligned} \quad (9)$$

A solution valid for small elastic effects and large values of ξ is sought in the following form.

$$A(\eta;\alpha_{1\xi};\alpha_{2\xi}) \sim A_0(\eta) + \alpha_{1\xi}A_{11}(\eta) + \alpha_{2\xi}A_{12}(\eta) + \ldots \quad (10)$$

where A represents any of the quantities F, V, c, and δ. Substituting (10) into (6), (8), and (9) results in

$$V_0' + F_0 = 0, \quad V_{11}' + F_{11} = 0, \quad V_{12}' + F_{12} = 0$$

$$F_0'' - (1+j\beta_1)V_0F_0' + \beta_2(1-F_0^2) = 0$$

TABLE 1. SKIN FRICTION AND DISPLACEMENT THICKNESS FUNCTIONS

j	p	V_w	c_0	δ_0	c_{11}	δ_{11}	c_{12}	δ_{12}
0	0.0	0	0.470	1.216	−0.273	0.435	—	—
0	0.2	0	0.802	0.890	−0.095	−0.032	—	—
0	0.4	0	0.977	0.775	0.158	−0.276	—	—
0	0.6	0	1.091	0.714	0.460	−0.475	—	—
0	0.8	0	1.172	0.675	0.791	−0.655	—	—
0	1.0	0	1.233	0.649	1.141	−0.826	—	—
1	0.2	0	1.001	0.645	0.052	0.058	0.058	−0.315
1	0.4	0	1.120	0.614	0.302	−0.153	0.046	−0.297
1	0.6	0	1.203	0.594	0.666	−0.339	0.040	−0.286
1	0.8	0	1.265	0.580	1.040	−0.513	0.036	−0.280
1	1.0	0	1.312	0.570	1.421	−0.681	0.033	−0.274
0	0	0.5	0.148	2.112	−0.109	0.688	—	—
0	0	−0.5	0.858	0.840	−0.517	0.197	—	—
0	0	−1.0	1.284	0.631	−0.814	0.070	—	—
0	1	0.5	0.970	0.782	0.925	−0.689	—	—
0	1	−0.5	1.542	0.542	1.290	−1.042	—	—
0	1	−1.0	1.889	0.459	1.373	−1.335	—	—
0	1	−2.0	2.670	0.342	1.389	−2.088	—	—
1	1	0.5	0.804	0.812	0.884	−0.567	−0.107	−0.209
1	1	−0.5	1.984	0.416	1.517	−1.136	0.343	−0.283
1	1	−1.0	2.765	0.318	1.221	−1.860	0.768	−0.265
1	1	−2.0	4.510	0.208	0.091	−3.691	1.752	−0.212

$$F_{11}' - (1 + j\beta_1)(V_0F_{11}' + V_{11}F_0') - 2\beta_2F_0F_{11} + (\beta_3 + j)F_0'^2 - (\beta_4 + j)V_0F_0''' - 2(\beta_3 - j)F_0F_0'' = 0$$

$$F_{12}'' - (1 + j\beta_1)(V_0F_{12}' + V_{12}F_0') - 2\beta_2F_0F_{12} - j(2F_0F_0'' + F_0'^2) = 0$$

$$c_0 = F_0'(0), \quad c_{11} = F_{11}'(0) - (\beta_4 + j)V_wF_0''(0), \quad c_{12} = F_{12}'(0)$$

$$\delta_0 = \int_0^\infty (1 - F_0)d\eta, \quad \delta_{11} = -\int_0^\infty F_{11}d\eta, \quad \delta_{12} = -\int_0^\infty F_{12}d\eta$$

$$F_0(0) = 0, \quad V_0(0) = V_w, \quad \lim_{\eta\to\infty} F_0(\eta) = 1$$

$$F_{11}(0) = F_{12}(0) = V_{11}(0) = V_{12}(0) = \lim_{\eta\to\infty} F_{11}(\eta) = \lim_{\eta\to\infty} F_{12}(\eta) = 0 \quad (11)$$

It can be seen that F_{12} vanishes for plane flow. The three systems of differential equations shown above were solved by the finite-difference method discussed by Richtmyer and Morton (1967).

RESULTS AND DISCUSSION

Some typical numerical results are shown in Figure 1 and Table 1. These results indicate that the effect of elasticity is influenced by various factors. For flow past a flat plate with no suction or injection ($j = 0$, $p = 0$, $V_w = 0$) increasing elasticity decreases skin friction and increases displacement thickness. For plane stagnation-point flow with no suction or injection ($j = 0$, $p = 1$, $V_w = 0$) the effect of elasticity is just the opposite. [These results have been established by previous investigators. See Peddieson (1971) for a list of references.] Table 1 shows the transition between these two limiting cases. Corresponding results are also given for cones. Figure 1 shows the second-order velocity profile function F_{11} for wedges with various vertex angles. It is clear that the second-order skin-friction coefficient c_{11} is increased by increasing the vertex angle while the second-order displacement thickness integral δ_{11} is decreased. The parameter c_{12} decreases with increasing p while δ_{12} increases.

Suction and injection also influence the effect of elasticity. For plane stagnation point flow the second-order skin-friction coefficient appears to be increased by suction and decreased by injection while the second-order displacement-thickness integral is increased by injection and decreased by suction. The numerical results for the skin-friction coefficient in this case are in agreement with

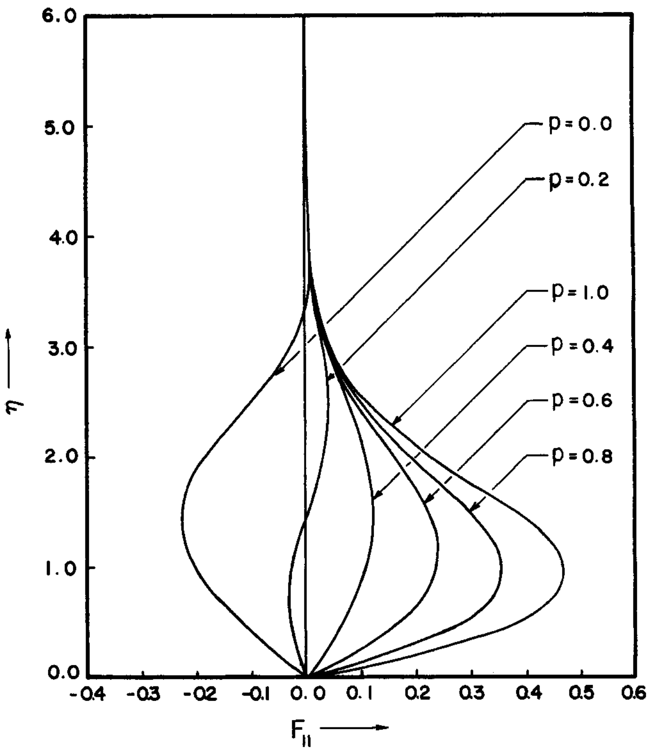

Fig. 1. Second-order velocity profile functions for wedges.

those given by Shrestha (1970) who used a slightly different constitutive equation. (Both yield the same boundary-layer equations for plane flow.) For flat plate flow both the second-order skin-friction coefficient and displacement-thickness integral are seen to be increased by injection and decreased by suction. For axisymmetric stagnation-point flow the behavior is more complicated as indicated by the last set of entries in Table 1.

CONCLUSION

In the present note the method of local similarity was used to obtain approximate solutions for laminar boundary-layer flows of second-order liquids on wedges and cones. Numerical results were presented to illustrate the effect of the value of the vertex angle of the body and the amount of suction or injection present at the body surface on the second-order skin-friction coefficient and displacement thickness integral.

NOTATION

L = characteristic length
n^* = dimensional normal coordinate (body surface is at $n^* = 0$)
n = $R^{1/2}n^*/L$
p = surface-speed index
R = $\rho LU/\eta_0$
s^* = dimensional tangential coordinate (body vertex is at $s^* = 0$)
s = s^*/L
u^* = dimensional tangential velocity
u = u^*/U
U = inviscid surface speed at $s^* = L$
v^* = dimensional normal velocity
v = $R^{1/2}v^*/U$
V_w = injection function
η_0 = viscosity
μ_1 = material constant
μ_2 = material constant
ρ = mass density

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Gas-Solid Chromatography with Nonuniform Adsorbent Surfaces

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Chromatography has been used both as an important application of adsorption and as a means for its study. Chromatographic moment-analysis methods have recently been employed (Grubner, 1968; Schneider and Smith, 1968; Padberg and Smith, 1968; Adrian and Smith, 1970) to evaluate parameters for a variety of transport, adsorption, and reaction processes in packed beds. That analytical relations can be derived affords a convenient opportunity to examine some implications of the details of the assumed process models (Dougharty, 1972).

It is well known that most practical adsorbent surfaces exhibit heterogeneity with respect to their interactions with adsorbing gases. It is the purpose of this note to explore briefly the consequences of surface nonuniformity for the response of packed chromatographic beds.

BEDS OF MIXED ADSORBENTS

Consider first the case of adsorption by a bed of homogeneously mixed, individually uniform, porous solid adsorbents. For spherical packing particles, uniformity over the column cross section, and linear adsorption/desorption rates, one has

$$\frac{\partial c_{a,j}}{\partial t} = k_{a,j}\,(c_{i,j} - c_{a,j}/K_{a,j}), \quad j = 1, \ldots, n \qquad (1)$$

$$\epsilon_{p,j}\frac{\partial c_{i,j}}{\partial t} = D_{i,j}\left(\frac{\partial^2 c_{i,j}}{\partial r^2} + \frac{2}{r}\frac{\partial c_{i,j}}{\partial r}\right) - \rho_{p,j}\frac{\partial c_{a,j}}{\partial t}, \quad j = 1, \ldots, n \qquad (2)$$

$$\epsilon_b\frac{\partial c_e}{\partial t} = -\,\epsilon_b\, v\frac{\partial c_e}{\partial z} + D_A\frac{\partial^2 c_e}{\partial z^2} - (1-\epsilon_b)\sum_{j=1}^{n}\omega_j\frac{3}{R_j}k^*_{m,j}\,(c_e - c_{i,j}|_{r=R_j}) \qquad (3)$$

The last term gives the locally volume-averaged rate of mass transfer from the interparticle volume to the intraparticle volume, per unit bed volume. ω_j can be expressed